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Recurrence Equations for the Computation of Correlations in the $1/r^2$ Quantum Many-Body System

P. J. Forrester[1]

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In a previous paper the two-particle distribution function and one-particle density matrix for the quantum many-body system with the $1/r^2$ pair potential have been expressed as limiting cases of Selberg correlation integrals. Recurrence equations are derived which allow rapid evaluation of these multidimensional integrals. The exact results for the two-particle distribution are compared with the harmonic approximation.

KEY WORDS: Selberg integrals; correlation functions; solvable models; $1/r^2$ quantum system.

1. INTRODUCTION

The Schrödinger operator

$$-\sum_{j=1}^{N} \frac{\partial^2}{\partial \theta_j^2} + g\left(\frac{\pi}{L}\right)^2 \sum_{1 \leqslant j < k \leqslant N} \frac{1}{\sin^2[\pi(\theta_k - \theta_j)/L]} \tag{1.1}$$

describing N particles on a line of length L interacting via the $1/r^2$ pair potential with periodic boundary conditions has the exact ground-state wave function[1]

$$\psi_0(\theta_1, \ldots, \theta_N) = \frac{1}{C_N} \prod_{1 \leqslant j < k \leqslant N} |\sin \pi(\theta_k - \theta_j)/L|^{\gamma/2} \tag{1.2a}$$

where

$$\gamma = 1 + (1 + 2g)^{1/2}, \qquad g \geqslant -1/2 \tag{1.2b}$$

[1] Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia.

and C_N denotes the normalization. As written, the state describes bosons; however, if multiplied by $\prod_{1 \leqslant j < k \leqslant N} \mathrm{sgn}(\theta_k - \theta_j)$, the Schrödinger equation is still satisfied and the state describes fermions.

The wave function (1.2) is of the BDJ type with pair potential

$$V(\theta - \theta') = -\log |\sin[\pi(\theta - \theta')/L]| \tag{1.3}$$

and thus corresponds to the Boltzmann factor of the one-component log-potential Coulomb gas confined to a circle of circumference length L. It is known[2] that (1.2a) is the unique BDJ wave function which correctly describes the low-energy excitations

$$\Delta(k) \sim \hbar c k \qquad \text{as} \quad k \to 0 \tag{1.4}$$

of an arbitrary one-component, one-dimensional quantum fluid. In this interpretation

$$\gamma = 2mc/\pi\rho\hbar \tag{1.5}$$

where m denotes the mass of the particles.

When γ is even, the two-particle distribution function (which is the same for both bosons and fermions)

$$h_N(\theta - \theta') := N(N-1) \int_0^L d\theta_3 \cdots \int_0^L d\theta_N (\psi_0(\theta, \theta', \theta_3, ..., \theta_N))^2 \tag{1.6}$$

is a trigonometric polynomial in $\cos[2\pi(\theta - \theta')/L]$ of order $\gamma(N-2)/2$. Similarly, when $\gamma/2$ is even (odd), the one-body density matrix

$$\rho_N^{(s)}(\theta - \theta') = N \left(\prod_{l=2}^{N} \int_0^L d\theta_l \right) \psi_0(\theta, \theta_2, ..., \theta_N)\, \psi_0(\theta', \theta_2, ..., \theta_N) \tag{1.7}$$

for $(s) = $ (f)ermions and $(s) = $ (b)osons, respectively, is a trigonometric polynomial in $\cos[2\pi(\theta - \theta')/L]$.

A study of the integrals (1.6) and (1.7) has been made recently in ref. 3. The first step in this work was to note that the integrals are limiting cases of the so-called Selberg correlation integrals. If we denote

$$D_{\lambda_1, \lambda_2, \lambda}(t_1, ..., t_N) := \prod_{l=1}^{N} t_l^{\lambda_1 - 1}(1 - t_l)^{\lambda_2 - 1} \prod_{1 \leqslant j < k \leqslant N} |t_k - t_j|^{\lambda} \tag{1.8}$$

then the (normalized) Selberg correlation integrals are defined as

$$\hat{S}_{N,m}(\lambda_1, \lambda_2, \lambda; x_1, ..., x_m)$$
$$:= \frac{1}{C_{N,m}} \left(\prod_{l=1}^{N} \cdot \int_0^1 dt_l \prod_{p=1}^{m} (t_l - x_p) \right) D_{\lambda_1, \lambda_2, \lambda}(t_1, ..., t_N) \tag{1.9a}$$

where

$$C_{N,m}=\left(\prod_{l=1}^{N}\int_0^1 dt_l\, t_l^m\right) D_{\lambda_1,\lambda_2,\lambda}(t_1,\ldots,t_N) \tag{1.9b}$$

In terms of these integrals it was shown that

$$\begin{aligned} h_N(\theta-\theta') &= \frac{N(N-1)}{L^2}\left[2\sin\left(\frac{\pi(\theta-\theta')}{L}\right)\right]^{\gamma} \\ &\quad \times e^{-\pi i\eta\gamma(\theta-\theta')(N-2)/N}\frac{[\gamma(N-2)/2]!\,[(\gamma/2)!]^2}{(\gamma N/2)!} \\ &\quad \times \hat{S}_{N-2,\gamma}\left(-\frac{\gamma(N-1)}{2},\gamma+1,\gamma;e^{2\pi i\eta(\theta-\theta')/N},\ldots,e^{2\pi i\eta(\theta-\theta')/N}\right) \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \rho_N^{(s)}(\theta-\theta') &= \eta e^{-\pi i\eta(\gamma/2)(\theta-\theta')}\frac{(\gamma/2)!\,(\gamma N/2)!}{[\gamma(N+1)/2]!} \\ &\quad \times \hat{S}_{N,\gamma}\left(-\frac{N\gamma}{2},\frac{\gamma}{2}+1,\gamma;e^{2\pi i\eta(\theta-\theta')/N},\ldots,e^{2\pi i\eta(\theta-\theta')/N}\right) \end{aligned} \tag{1.11}$$

where

$$\eta := N/L \tag{1.12}$$

The significance of these formulas is that the Selberg correlation integrals have been evaluated in terms of a new class of multivariable hypergeometric functions based on Jack symmetric polynomials.[4] In the special cases (1.10) and (1.11), when all the arguments are equal, the explicit power series expansion of the hypergeometric function, and thus a formula for the coefficients of the trigonometric polynomials representing $h_N(\theta-\theta')$ and $\rho_N^{(s)}(\theta-\theta')$, can be given.[3] Unfortunately, the resulting formula is unsatisfactory from the computational viewpoint, as the coefficient of x^k ($x=e^{2\pi i\eta(\theta-\theta')/N}$) involves a sum over all partitions of k into γ parts, and thus the number of operations required to compute each coefficient increases factorially in both k and γ.

The main purpose of this paper, which is contained in Section 2, is to derive and implement recurrence equations which uniquely define the Selberg correlation integrals (1.9) in the case $x_1=x_2=\cdots=x_m=x$. These recurrences generate the coefficients of the power series after only $O((\gamma N)^2)$ operations. The method we use is adapted from the work of Aomoto[5,6] and Edelman.[7]

In Section 3 the two-particle distribution function (1.6) is computed in the harmonic approximation and the exact and approximate calculations are compared.

2. THE RECURRENCES

2.1. Derivation

Let us first introduce some notation. Define

$$I_p^{(\alpha)}[g]=\frac{1}{c}\int_0^1 dt_1\cdots\int_0^1 dt_N\, gF_{\lambda_1,\lambda_2,\gamma}(t_1,..., t_N; \alpha, p, x) \tag{2.1a}$$

where

$$\begin{aligned}F_{\lambda_1,\lambda_2,\gamma}&(t_1,..., t_N; \alpha, p, x)\\ &:=(t_1-x)^{\alpha}\cdots(t_p-x)^{\alpha}\,(t_{p+1}-x)^{\alpha-1}\cdots(t_N-x)^{\alpha-1}\\ &\quad\times D_{\lambda_1,\lambda_2,\gamma}(t_1,..., t_N)\end{aligned} \tag{2.1b}$$

and c is independent of x, and adopt the abbreviation

$$I_p^{(\alpha)}[1]=I_p^{(\alpha)} \tag{2.2}$$

Comparison of (2.1) and (1.9) shows that for the choice of c such that when $x=0$, $I_p^{(\alpha)}=1$, we have

$$I_N^{(\gamma)}=\hat{S}_{N,\gamma}(\lambda_1, \lambda_2, \lambda; x,..., x) \tag{2.3}$$

The recurrences given in the following result uniquely specify $I_N^{(\gamma)}$, and provide a practical method for its evaluation.

Proposition 1. We have

$$I_N^{(\alpha-1)}=I_0^{(\alpha)} \tag{2.4}$$

and

$$I_{p+1}^{(\alpha)}=-(Ax+B)\,I_p^{(\alpha)}+Cx(x-1)\frac{d}{dx}I_p^{(\alpha)}+Dx(x-1)\,I_{p-1}^{(\alpha)} \tag{2.5}$$

where

$$A=[\lambda_1+\lambda_2+\gamma(N-p-1)+2(\alpha-1)]/E \tag{2.6a}$$

$$B=-[\lambda_1-1+\alpha+(\gamma/2)(N-p-1)]/E \tag{2.6b}$$

$$C=1/[(N-p)\,E] \tag{2.6c}$$

$$D=p[\gamma/2+\alpha/(N-p)]/E \tag{2.6d}$$

with

$$E = \lambda_1 + \lambda_2 + \gamma(N - p/2 - 1) + \alpha - 1 \tag{2.6e}$$

The recurrence (2.4) follows immediately from the definitions (2.1) and (2.2). The proof of (2.5) can conveniently be broken up into a number of lemmas.

Lemma 1:

$$I_p^{(\alpha)}[t_{p+1}] = I_{p+1}^{(\alpha)} + xI_p^{(\alpha)} \tag{2.7}$$

$$I_p^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_k}\right] = \begin{cases} \dfrac{-x}{2} I_{p-1}^{(\alpha)}, & k \leqslant p \quad (2.8a) \\[2mm] \dfrac{1}{2} I_p^{(\alpha)}, & k > p+1 \quad (2.8b) \end{cases}$$

$$I_{p+1}^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_k}\right] = \begin{cases} \dfrac{1}{2} I_{p+1}^{(\alpha)}, & k < p+1 \\[2mm] I_{p+1}^{(\alpha)} + \dfrac{x}{2} I_p^{(\alpha)}, & k > p+1 \end{cases} \tag{2.9}$$

and

$$\frac{d}{dx} I_p^{(\alpha)} = -p\alpha I_{p-1}^{(\alpha)} - (N-p)(\alpha - 1)\, I_p^{(\alpha)}\left[\frac{1}{t_{p+1} - x}\right] \tag{2.10}$$

Proof. Equation (2.7) follows from the definition of $I_p^{(\alpha)}$ and the simple manipulation $t_{p+1} = (t_{p+1} - x) + x$. To derive (2.8b), interchange t_{p+1} and t_k to deduce that

$$I_p^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_k}\right] = I_p^{(\alpha)}\left[\frac{t_k}{t_{p+1} - t_k}\right] \tag{2.11a}$$

Taking the arithmetic mean of both sides gives the required result.

To derive (2.8a), note

$$\begin{aligned} I_p^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_k}\right] &= I_p^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_p}\right] \\ &= I_{p-1}^{(\alpha)}\left[\frac{t_p t_{p+1}}{t_{p+1} - t_p}\right] - xI_{p-1}^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_p}\right] \end{aligned} \tag{2.11b}$$

The first term on the right-hand side of the second equality vanishes, as the integrand is antisymmetric under the interchange $t_p \leftrightarrow t_{p+1}$. For the second term we use (2.8b) and thus obtain (2.8a). Similar arguments suffice to establish (2.9). To derive (2.10), we note

$$\frac{d}{dx} I_p^{(\alpha)} = -\alpha \sum_{j=1}^{p} I_p^{(\alpha)}\left[\frac{1}{t_j - x}\right] - (\alpha - 1) \sum_{j=p+1}^{N} I_p^{(\alpha)}\left[\frac{1}{t_j - x}\right]$$
$$= -p\alpha I_{p-1}^{(\alpha)} - (\alpha - 1) \sum_{j=p+1}^{N} I_p^{(\alpha)}\left[\frac{1}{t_j - x}\right] \tag{2.12}$$

The required result now follows by renaming the variables of integration in the last sum. ▮

Lemma 2. We have

$$(\lambda_2 - 1) I_p^{(\alpha)}\left[\frac{1}{1 - t_{p+1}}\right] = \left[\lambda_1 + \lambda_2 - 2 + \alpha + \frac{\gamma}{2}(N - p - 1)\right] I_p^{(\alpha)}$$
$$- \frac{x}{N-p}\frac{d}{dx} I_p^{(\alpha)} - xp\left(\frac{\alpha}{N-p} + \frac{\gamma}{2}\right) I_{p-1}^{(\alpha)} \tag{2.13}$$

Proof. Let $F = F_{\lambda_1, \lambda_2, \gamma}(t_1, ..., t_N; \alpha, p, x)$ be given by (2.1b). By the fundamental theorem of calculus, for all λ_1 and λ_2 large enough so that $t_{p+1}F$ vanishes at $t_{p+1} = 0$ and 1,

$$\int_0^1 dt_1 \cdots \int_0^1 dt_N \frac{\partial}{\partial t_{p+1}}(t_{p+1}F) = 0 \tag{2.14}$$

Performing the partial derivative and simple manipulation gives

$$\lambda_1 I_p^{(\alpha)} + (\lambda_2 - 1) I_p^{(\alpha)} - (\lambda_2 - 1) I_p^{(\alpha)}\left[\frac{1}{1 - t_{p+1}}\right]$$
$$+ \gamma \sum_{\substack{k=1 \\ k \neq p+1}}^{N} I_p^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_k}\right] + (\alpha - 1) I_p^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - x}\right] = 0 \tag{2.15}$$

The result (2.13) now follows by use of Lemma 1 and rearrangement. ▮

Lemma 3. We have

$$-\left[\lambda_1 + \lambda_2 + \gamma\left(N - \frac{p}{2} - 1\right) + \alpha - 1\right] I_{p+1}^{(\alpha)}$$
$$= \{(\lambda_1 + 1) x + (\lambda_2 - 1)(x + 1) + x\gamma(N - p - 1) + 2x(\alpha - 1)\} I_p^{(\alpha)}$$
$$- \frac{x^2}{N-p}\frac{d}{dx} I_p^{(\alpha)} - \left(\frac{\gamma p x^2}{2} + \frac{p\alpha x^2}{N-p}\right) I_{p-1}^{(\alpha)}$$
$$- (\lambda_2 - 1) I_p^{(\alpha)}\left[\frac{1}{1 - t_{p+1}}\right] \tag{2.16}$$

Proof. Proceeding analogously to (2.14), we note that from the fundamental theorem of calculus,

$$\int_0^1 dt_1 \cdots \int_0^1 dt_N \frac{\partial}{\partial t_{p+1}} \{(t_{p+1})^2 F\} = 0 \tag{2.17}$$

Performing the partial derivative gives

$$(\lambda_1 + 1) I_p^{(\alpha)}[t_{p+1}] - (\lambda_2 - 1) I_p^{(\alpha)}\left[\frac{(t_{p+1})^2}{1 - t_{p+1}}\right] + \gamma \sum_{\substack{k=1 \\ k \neq p+1}}^{N} I_p^{(\alpha)}\left[\frac{(t_{p+1})^2}{t_{p+1} - t_k}\right]$$

$$+ (\alpha - 1) I_p^{(\alpha)}\left[\frac{(t_{p+1})^2}{t_{p+1} - x}\right] = 0 \tag{2.18}$$

Now,

$$I_p^{(\alpha)}\left[\frac{(t_{p+1})^2}{1 - t_{p+1}}\right] = I_p^{(\alpha)}\left[\frac{(t_{p+1} - 1)(t_{p+1} + 1) + 1}{1 - t_{p+1}}\right]$$

$$= -I_p^{(\alpha)} - I_p^{(\alpha)}[t_{p+1}] + I_p^{(\alpha)}\left[\frac{1}{1 - t_{p+1}}\right] \tag{2.19}$$

$$I_p^{(\alpha)}\left[\frac{(t_{p+1})^2}{t_{p+1} - t_k}\right] = I_p^{(\alpha)}\left[\frac{t_{p+1}(t_{p+1} - x + x)}{t_{p+1} - t_k}\right]$$

$$= I_{p+1}^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_k}\right] + x I_p^{(\alpha)}\left[\frac{t_{p+1}}{t_{p+1} - t_k}\right] \tag{2.20}$$

and

$$I_p^{(\alpha)}\left[\frac{(t_{p+1})^2}{t_{p+1} - x}\right] = I_p^{(\alpha)}\left[\frac{(t_{p+1} - x)(t_{p+1} + x) + x^2}{t_{p+1} - x}\right]$$

$$= x I_p^{(\alpha)} + I_p^{(\alpha)}[t_{p+1}] + x^2 I_p^{(\alpha)}\left[\frac{1}{t_{p+1} - x}\right] \tag{2.21}$$

The terms on the right-hand side of (2.19)–(2.21) and the first term in (2.18) can be further simplified using Lemma 1. The formula (2.16) follows after substituting the resulting expressions back in (2.18). ∎

The recurrence relation (2.5) of Proposition 1 follows immediately from Lemmas 2 and 3 by substituting (2.13) in (2.16) and rearranging.

2.2. Implementation

From the definitions (2.2) and (2.1) we note that $I_p^{(\alpha)}$ is a polynomial in x of degree $(\alpha - 1) N + p$. From Proposition 1, these polynomials can be calculated recursively, starting with the initial condition $I_0^{(1)} = 1$ and using (2.5) to calculate $I_1^{(1)}, I_2^{(1)}, \ldots, I_N^{(1)}$ in order. From (2.4), $I_N^{(1)} = I_0^{(2)}$, so now $I_1^{(2)}, \ldots, I_N^{(2)}$ can be calculated from (2.5), etc. $I_N^{(\gamma)}$ can thus be calculated after γN applications of (2.5), which could be carried out on a computer algebra package.

Alternatively, Eqs. (2.4) and (2.5) can be used to derive recursive equations for the coefficients in the polynomial

$$I_p^{(\alpha)} = \sum_{k=0}^{(\alpha-1)N+p} a^{(\alpha)}(k, p)\, x^k \tag{2.22}$$

Yet another method is to use (2.4) and (2.5) to compute $I_p^{(\alpha)}$ for a specific value of x (x_0, say). This requires calculating

$$\frac{d^r}{dx^r} I_p^{(\alpha)} \qquad (r = 1, 2, \ldots, (\alpha-1)N + p) \tag{2.23}$$

at $x = x_0$, which can be done recursively from (2.5) using Leibnitz's rule. Both these methods require $O((\gamma N)^2)$ operations.

In practice we have implemented the two methods of the above paragraph to calculate $I_N^{(\alpha)}$ and thus, from (2.3), $\hat{S}_{N,\gamma}$. From (1.10) we must calculate $\hat{S}_{N,\gamma}$ with $N \mapsto N-2$, $\lambda_1 = -\gamma(N-1)/2$, and $\lambda_2 = \gamma + 1$ in order to calculate the two-particle distribution function, while from (1.11) we see that the calculation of the density matrix requires $\hat{S}_{N,\gamma}$ with $\lambda_1 = -N\gamma/2$ and $\lambda_2 = \gamma/2 + 1$. As an illustration, for $N = 14$ and $\gamma = 8$ and 12 we have computed $h_N(\theta)$ in this way and plotted the corresponding graphs in Fig. 1.

With double-precision arithmetic, catastrophic cancellation restricted the calculations to $\gamma N \leqslant 200$. This problem can be eliminated by the use of

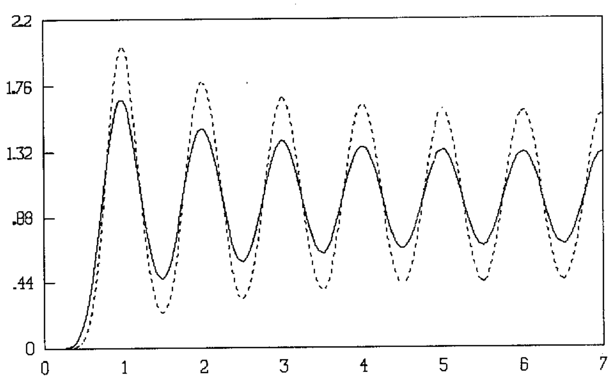

Fig. 1. Plot of $h_N(\theta)$ with $\eta = 1$ for $N = 14$ and $\gamma = 8$ (full line), $\gamma = 12$ (dashed line).

a high-precision computing package. Also, it was found that the large-N behavior of $h_N(\theta)$ was accurately given by $N=12$ for $\theta<3$, although the accuracy deteriorates slightly as γ is increased.

3. HARMONIC APPROXIMATION

Krivnov and Ocvhinnikov[8] applied a harmonic approximation to the Hamiltonian (1.1) and gave the corresponding expression for the two-particle distribution function. In this section we will compute the two-particle distribution function in the harmonic approximation directly from the ground-state wave function (1.2a) (see also ref. 9) and compare it to the exact evaluation of the previous section.

Let us consider (1.2a) as the Boltzmann factor for the classical gas with pair potential (1.3). Since the potential is repulsive, in the low-temperature, $\gamma \to 0$ limit the particles tend to distribute themselves at equal spacings L/N apart. Thus, if we make the ordering

$$\theta_1<\theta_2<\cdots<\theta_N \tag{3.1}$$

then the ground-state configuration is

$$\theta_j=v+(j-1)\,L/N, \qquad \text{where} \quad 0\leqslant v\leqslant L/N \tag{3.2}$$

In the harmonic approximation the total potential energy of the classical gas is expanded up to second order about the equilibrium points (3.2), and the two-particle distribution function is written as

$$h_N^{(\text{ha})}(\theta)=(\eta/Q)\sum_{p=2}^{N}\int_{-\infty}^{\infty}d\theta_1\cdots\int_{-\infty}^{\infty}d\theta_N \\ \times\delta(\theta_1)\,\delta(\theta_p-\theta)\exp(-\gamma\underset{\sim}{\theta}^T H\underset{\sim}{\theta}/2) \tag{3.3a}$$

where

$$Q=\int_{-\infty}^{\infty}d\theta_1\cdots\int_{-\infty}^{\infty}d\theta_N\,\delta(\theta_1)\exp(-\gamma\underset{\sim}{\theta}^T H\underset{\sim}{\theta}/2) \tag{3.3b}$$

$$\underset{\sim}{\theta}=(\theta_j-(j-1)\,L/N)_{j=1,\dots,N} \qquad \text{and} \qquad H:=[a_{jk}]_{j,k=1,\dots,N} \tag{3.3c}$$

with

$$a_{jj}=\left(\frac{\pi}{L}\right)^2\sum_{n=1}^{N-1}\frac{1}{\sin^2\pi n/N}, \\ a_{jk}=-\left(\frac{\pi}{L}\right)^2\frac{1}{\sin^2\pi(j-k)/N} \qquad (j\neq k) \tag{3.3d}$$

The integrals (3.3a) and (3.3b) can be transformed by (i) changing variables $\theta_j - (j-1)\,L/N \mapsto \phi_j$, (ii) using the integral representation

$$\delta(x) = \int_{-\infty}^{\infty} e^{2\pi i x t}\, dt \tag{3.4}$$

and (iii) changing variables $\underset{\sim}{\phi} = U\underset{\sim}{y}$, where

$$U = [e^{2\pi ijk/N}]_{j,k=0,\ldots,N-1} \qquad \text{and} \qquad \underset{\sim}{y} = (y_j)_{j=0,\ldots,N-1} \tag{3.5}$$

This gives

$$\begin{aligned} h_N^{(\mathrm{ha})}(\theta) = (\eta/Q) \sum_{p=1}^{N-1} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \exp(-2\pi i\theta t_2) \\ \times \int_{-\infty}^{\infty} dy_0 \exp[2\pi i y_0 (t_1 + t_2)/\sqrt{N}] \\ \times \left(\prod_{j=2}^{N-1} \int_{-\infty}^{\infty} d\,\mathrm{Re}(y_j) \int_{-\infty}^{\infty} d\,\mathrm{Im}(y_j) \exp(-\gamma\lambda_j\, |y_j|^2/2) \right. \\ \left. \times \exp\{2\pi i y_j [t_1 + t_2 \exp(2\pi i p j/N)]/\sqrt{N}\} \right)^{1/2} \end{aligned} \tag{3.6a}$$

where

$$\begin{aligned} Q = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy_0\, e^{2\pi i t y_0/\sqrt{N}} \left(\prod_{j=1}^{N-1} \int_{-\infty}^{\infty} d\,\mathrm{Re}(y_j) \int_{-\infty}^{\infty} d\,\mathrm{Im}(y_j) \right. \\ \left. \times e^{-\gamma\lambda_j\, |y_j|^2/2} e^{2\pi i t y_j/\sqrt{N}} \right)^{1/2} \end{aligned} \tag{3.6b}$$

and

$$\lambda_j = \left(\frac{\pi}{L}\right)^2 \sum_{n=1}^{N-1} \frac{1 - \cos 2\pi j n/N}{\sin^2 \pi n/N} \tag{3.6c}$$

The integration over y_0 gives $\sqrt{N}\,\delta(t_1 + t_2)$ in (3.6a) and $\sqrt{N}\,\delta(t_1)$ in (3.6b), so the integration over t_1 can be performed immediately. The integrations over $y_1, \ldots, y_{N-1}$ are now all independent Gaussians, which can be computed by completing the square. The remaining integration over t_2 is also of the Gaussian type. We thus obtain

$$h_N^{(\mathrm{ha})}(\theta) = \eta^2 \sum_{p=1}^{N-1} \left(\frac{\gamma}{4\pi f(p)}\right)^{1/2} e^{-\gamma(p - \eta x)^2/(4\pi^2 f(p))} \tag{3.7a}$$

where

$$f(p)=\frac{\eta^2}{2N}\sum_{j=1}^{N-1}\frac{1-\cos 2\pi pj/N}{\lambda_j} \tag{3.7b}$$

In the limit $N\to\infty$, j/N fixed

$$\lambda_j \to \eta^2 \sum_{n=1}^{\infty}\frac{1-\cos 2\pi jn/N}{n^2}$$
$$=(\pi\eta)^2\left[\left|\frac{j}{N}\right|-\left(\frac{j}{N}\right)^2\right], \qquad \left|\frac{j}{N}\right|<\frac{1}{2} \tag{3.8}$$

(note that $\lambda_j=\lambda_{j+N}$). Hence, from (3.7b),

$$f(p)\to\frac{1}{\pi^2}\int_0^{1/2}\frac{1-\cos 2\pi pt}{t-t^2}\,dt \tag{3.9}$$

and with $h^{(\text{ha})}(\theta)$ denoting $\lim_{N\to\infty} h_N^{(\text{ha})}(\theta)$, from (3.7a),

$$h^{(\text{ha})}(\theta)=\eta^2\sum_{\substack{p=-\infty\\ p\neq 0}}^{\infty}\left(\frac{\gamma}{4\pi f(p)}\right)^{1/2} e^{-\gamma(p-\eta\theta)^2/(4\pi^2 f(p))} \tag{3.10}$$

The formula (3.10) is particularly easy to evaluate. We find that $h^{(\text{ha})}(\theta)$ is a very good approximation to $h_N^{(\theta)}$ for $\gamma\geqslant 20$, with the corresponding graphs being difficult to distinguish, as is illustrated in Fig. 2.

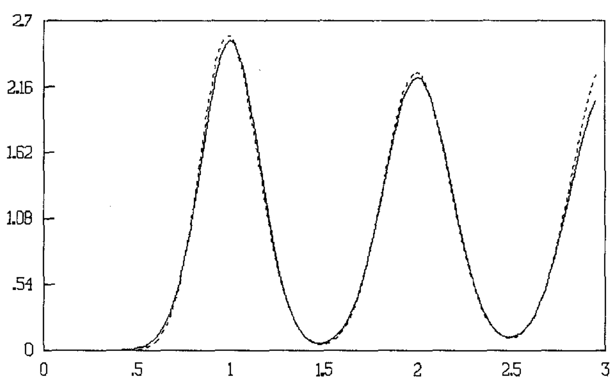

Fig. 2. Plot of $h_N(\theta)$ (dashed line) and $h^{(\text{ha})}(\theta)$ (full line) with $\eta=1$, $N=6$ for $\gamma=20$.

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